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GAME PROBLEM OF IMPULSE ENCOUNTER WITH AN OPPONENT LIMITED IN ENERGY

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We examine the following problem. Two material points $m_{1,2}$ (the first and second players) [1, 2] move in a three-dimensional space under the action of position forces $F_{1,2} = -\omega^2 m_{1,2} r_{1,2}$, of attraction to a fixed center, and of control forces $f_{1,2}$ directed arbitrarily. We assume that the number $\omega^2 \geqslant 0$, that the masses $m_{1,2}$ are constant, and that $r_{1,2}$ are the radius vectors of the points relative to the center. The force $f_1 = m_1 u$ is bounded in total momentum [3, 5], while the force $f_2 = -m_2 v$ is limited in energy

$$\nu_0^2 - \int_0^\tau v^2 dt = \nu^2(\tau) \geqslant 0, \qquad \nu = \sqrt{\nu^2} \geqslant 0$$

The task of the first (second) player is to minimize (maximize) the time of encounter with the opponent at a distance $R = |r_1 - r_2| \geqslant 0$. The paper is similar to [5, 6] in the methods used to solve the problem.

1. After norming, leading to the equality $\omega^2 = 1$ (for $\omega^2 > 0$), the equations of relative motion take the form

$$x^{\cdot} = y_\alpha, \quad y_\alpha^{\cdot} = -x + y_\beta^2 / x + u_\alpha + v_\alpha \tag{1.1}$$
$$y_\beta^{\cdot} = -y_\alpha y_\beta / x + u_\beta + v_\beta, \quad \mu^{\cdot} = -|u|, \quad (\nu^2)^{\cdot} = -v^2$$

in the variables $x_1 = r_1 - r_2$, $y_1 = r_1^{\cdot} - r_2^{\cdot}$, $x = |x|$, y_α, y_β. The vector j_α ($|j_{\alpha,\beta,\gamma}| = 1$) is chosen along vector x_1, the vector j_β along the transverse component y_β^1 of vector y_1, and the vector j_γ completes the triple. If $y_\beta^1 = 0$, the pair of unit vectors $j_{\beta,\gamma}$ are directed arbitrarily in a plane normal to j_α. The lower indices indicate projections with respect to the unit vectors. When $y_\beta = 0$ the third equation of system (1.1) becomes

$$y_\beta^{\cdot} = [(u_\beta + v_\beta)^2 + (u_\gamma + v_\gamma)^2]^{1/2}$$

The motion governed by system (1.1) takes place under the phase constraints $\mu \geqslant 0$, $\nu^2 \geqslant 0$. The impulse control $u = \mu_1 \delta$ of the first player leads to the position $w[x, y_\alpha, y_\beta, \mu, \nu]$ as a result of the impulse

$$w^{(1)}[x,\ y_\alpha^{(1)},\ y_\beta^{(1)},\ \mu^{(1)},\ \nu]$$
$$y_\alpha^{(1)} = y_\alpha + \mu_{1,\alpha}, \qquad y_\beta^{(1)} = [(y_\beta + \mu_{1,\beta})^2 + \mu_1^2\gamma]^{1/2}, \ \mu^{(1)} = \mu - |\mu_1| \geqslant 0$$

The definition of admissible pairs $u(w, v)$, $v(w)$ was given in [5].

We agree to some notation. The number x and the vector j_α, under a shift along a trajectory of system (1.1) with $u+v=0$, are transformed by instant t to the number $x_t=[(xb_t+y_\alpha a_t)^2+y_\beta{}^2a_t{}^2]^{1/2}$, $a_t=\sin t$, $b_t=\cos t$ and to the vector $j_{\alpha,t}=[(xb_t+y_\alpha a_t)\,j_\alpha+y_\beta a_t j_\beta]\,/\,x_t$. In $(w,\ t)$-space we shall introduce below, in different ways, the function $\zeta\,(w,\ t)$ and the function $t_\zeta\,(w)$, the latter being the smallest positive zero of function ζ. The replacement of the subscript t by ζ signifies in all cases that $t=t_\zeta$. For example, $a_\zeta=\sin t_\zeta\,(w)$, $x_\zeta=x_{t=t_\zeta}$.

If a domain $D_{j,t}$ is specified in some way, say

$$D_{1,\,t}\,[R-\nu\sqrt{t\,/\,2-(\sin 2t)\,/\,4}\geqslant 0,\ t\in(0,\ \pi\,/\,2]]$$

and if the numbers ν, t satisfy both bounds simultaneously, then the domain

$$D_{1,\,\zeta}\,[R-\nu\sqrt{t_\zeta\,/\,2-(\sin 2t_\zeta)\,/\,4}\geqslant 0,\qquad t_\zeta\in(0,\ \pi\,/\,2]]$$

indicates a boundary in w-space.

Suppose that at $t=0$ the first player has realized the impulse $u=\mu_1\delta$ and that for $t>0$ employs a finite u, while the second player applies v; motion for $t>0$ is a result of impulse $w^{(1)}$. The notation $t_\zeta{}^\cdot\,(w^{(1)},u,\ v)$ refers to the right derivative of function t_ζ along a motion due to system (1.1).

2. We split up the construction of the solution into a series of auxiliary problems.

Problem 2.1. Let $\mu=0$. Find the control $v^\circ\,(w,\ \mu=0)$ and the time $t_\zeta\,(w,\ \mu=0)$ of slow-action on the set $M\,[x=R>0]$. The restriction $R>0$ is necessary for Problem 2.1 to make sense; we seek the problem's solution in the domain $W_1\,[x>R]$. An integration of the characteristics [1] by the scheme in [6] in the fixed system of coordinates x_1, y_1, leads to the following conclusions.

2.1.1. Let $w\in W_1$ be some position for which exists a function $t_\zeta\,(w,\ \mu=0)$ continuously differentiable at this position and let the optimal trajectory issuing from w hit set M at point x_1; then for $t_\zeta<\pi$ the control v° has the constant direction $-\,x_1\,/\,|\,x_1\,|$ along the trajectory, while its absolute value along the trajectory changes by the law $|\,v^\circ\,|=c\,(w)\,\sin\,(t_\zeta-t)$, where $c\,(w)\geqslant 0$ depends solely upon the trajectory (characteristic).

2.1.2. A formal application of the necessary conditions leads to the conclusion that the derivatives $\partial t_\zeta\,/\,\partial\nu^2>0$ must be constant along the characteristics of the main equation [1]. On the other hand, we have the obvious equality $\partial t_\zeta\,/\,\partial\nu^2\,|_{w\in M}=0$. This compels us to assume that the derivatives mentioned are discontinuous. The difficulty is overcome by a trivial argument: the slow-action control v° consumes the whole energy resource ν^2. The last assumption leads to the equalities

$$\int_{t_\zeta}^{0} c^2\sin^2(t_\zeta-t)\,dt=\nu^2 \tag{2.1}$$

$$c\,(w)=\nu/d_\zeta,\qquad d_t=\sqrt{t/2-(\sin 2t)/4}$$

We fix the $j_{\alpha,\beta,\gamma}$ corresponding to some position w as a fixed triple. According to 2.1.1 the control v° has the form

$$v^\circ=(v_\alpha{}^\circ j_\alpha+v_\beta{}^\circ j_\beta)\,c\sin\,(t_\zeta-t)$$

along the characterisitic. Here $t>0$ is the time of motion along the characteristic, while the function t_ζ and the vector

$$v_\alpha{}^\circ j_\alpha + v_\beta{}^\circ j_\beta = -x_1 / |x_1| = x_1(t_\zeta) / |x_1(t_\zeta)| \qquad (2.2)$$

are yet to be determined but are constant along the characteristic. The components $x_{\alpha,\zeta}$, $x_{\beta,\zeta}$ of vector $x_1(t_\zeta)$ correspond by Cauchy's formula to the equations

$$x_{\alpha,\zeta} = xb_\zeta + y_\alpha a_\zeta + v_\alpha{}^\circ c \int_0^{t_\zeta} \sin^2(t_\zeta - t)\, dt \qquad (2.3)$$

$$x_{\beta,\zeta} = y_\beta a_\zeta + v_\beta{}^\circ c \int_0^{t_\zeta} \sin(t_\zeta - t)\, dt$$

Equalities (2.1) – (2.3) and the condition for the continuity of t_ζ on set M permit us to seek t_ζ as the smallest zero of the function $\zeta = R - \nu d_t - x_t$. The function $\zeta(w, t)$ is obtained by superposing the monotonically decreasing function $-\nu d_t$ on the t-periodic function $R - x_t$. It is easy to verify that not more than two points $\tau_1 < \tau_2$ of isolated maximum of function ζ are present on the period $t \in (0, \pi)$ of function $R - x_t$; moreover, the bound $\zeta_1 \equiv \zeta(w, \tau_1) > \zeta_2 \equiv \zeta(w, \tau_2)$ is valid for $\nu > 0$. When $y_\alpha = 0$ a further case is possible, namely, $\zeta_{t=0} = 0$, $\zeta''_{t=0} < 0$, i.e. the "isolated for $t > 0$" maximum at $t = 0$. However, the bound $\zeta_1 > \zeta_2$ for all those positions at which $\tau_2 < \pi$ exists guarantees the continuity [5] of the function $t_\zeta(w, \mu = 0)$ in the region $t_\zeta \in (0, \pi)$. This important fact, together with the equality $\zeta'_{t=0} = y_\alpha$ enables us to assert that t_ζ is the first instant at which the trajectory w° from position w hits set M. Trajectory w° is generated by the equation

$$v^\circ = a_\zeta (\nu / d_\zeta) j_{\alpha,\zeta}, \quad w \in W^{\circ,\circ} \in [x_\zeta > 0]$$

$$v^\circ = a_\zeta (\nu / d_\zeta) j_s, \quad w \in W^{\circ,\circ} \cap [x_\zeta = 0]$$

$$s = +\nu d_\zeta / \sqrt{(-xb_\zeta + y_\alpha a_\zeta)^2}, \quad j_s = s j_\alpha + \sqrt{1 - s^2}\, j_\beta$$

where $W^{\circ,\circ}[\zeta_1 \geqslant 0]$ denotes the domain admitting the zero t_ζ. The control v° satisfies all the necessary conditions in the domain $w \in W^{\circ,\circ} \cap [x_\zeta > 0]$. It can be shown that the control v° realizes the equality

$$0 = \zeta_1^{\boldsymbol{\cdot}}(w, v^\circ) \ll \zeta_1^{\boldsymbol{\cdot}}(w, v)$$

at positions $w \in [t_\zeta = \tau_1]$. This equality and bound were the foundation of the construction of v° at the positions $w \in [x_\zeta = 0]$, since here the characterisitcs stick together and admit of a whole "sheaf" of optimal controls v° with a like time t_ζ. It is clear that the derivatives $\partial t_\zeta / \partial w$ do not exist on the set $[\tau_1 = t_\zeta]$. This remark is valid in all that follows. By analogy with [5, 6] we continue the control into the domain $W_{0,0} = [x > R] \setminus W^{\circ,\circ}$ by means of the equalities

$$v_0 = v^\circ(w, t_\zeta = \tau_1), \quad w \in W_{0,0} \cap [\zeta(w, 0) < \zeta_1] \equiv W_{1,0,0}$$

$$v_0 = 0, \quad w \in W_{0,0} \setminus W_{1,0,0}$$

Theorem 2.1. In the domain $W^{\circ,\circ}$ the control v° realizes the slow-action $t_\zeta(w, \mu = 0)$, while in the domain $W_{0,0}$ the control v_0 avoids a position on set M.

To prove the first assertion it is sufficient to verify the estimate $-1 = t^{\cdot}_{\zeta}(w, v^{\circ}) \geqslant t_{\zeta}^{\cdot}(w, v)$ for $w \in [x_{\zeta} > 0]$ and for those controls v (at the positions $w \in [x_{\zeta} = 0]$) which realize the finite derivative $t_{\zeta}^{\cdot}$ and to show that the relation

$$t_{\zeta}^{\cdot}(w \to [x_{\zeta} = 0], \quad v) \to -\infty$$

is valid for the rest of the nonoptimal controls. The second assertion follows from the estimate

$$(\max_t \zeta(w, \quad t \in [0, \pi / 2]))^{\cdot}_{v=v_0} \leqslant 0$$

This estimate turns into an equality for $w \in W_{1,0}$ and into a strict inequality for $w \in W_{0,0} \setminus W_{1,0}$.

Such are the results of solving Problem 2.1.

3. Past experience [5, 6] shows that the first player's optimal control is an impulse control; therefore, we formulate one more auxiliary problem for many positions.

Problem 3.1. Among the impulse controls $u = \mu_1\delta$, $|\mu_1| \leqslant \mu$ find the control u° corresponding to the equality

$$t_{\zeta}^{(1)}(w) = \min{}_u t_{\zeta}(w^{(1)}(w, \ u))$$

The solution of this problem reduces to simple operations by the implicit function theorem and has the form

$$\zeta = R - \nu d_t + \mu a_t - x_t \qquad (3.1)$$

If function ζ admits of a zero in the domain $D_{1,t}$ $[\xi \equiv R - \nu d_t \geqslant 0,\ t \in (0, \pi / 2]]$, then $t_{\zeta}^{(1)}(w) = t_{\zeta}$ (i.e. the minimum equals the smallest zero) and is realized on the vector $u^{\circ} = -\mu j_{\alpha,\zeta}$.

Problem 3.2. Find the impulse $u_0 = -\mu_{(0)}\delta$ realizing the estimate

$$t_{\zeta}(w^{(1)}(u_0)) \leqslant t_{\zeta}(w^{(1)}(\mu_1))$$

Solution. Among the impulses μ_1 which preserve the inclusion $w^{(1)} \in D_{1,\zeta}$ the impulses $u_{(0)} = mu^{\circ}$ $(0 \leqslant m \leqslant 1)$ solve Problem 3.2. The latter family has been defined to within a factor. The correspondence is verified on the basis of the relations

$$\zeta(w, t) \leqslant \zeta(w^{(1)}(\mu_1), t), \qquad \zeta(w, t_{\zeta}) = \zeta(w^{(1)}(u^{\circ}), \ t_{\zeta})$$

Problem 3.3. Among the family $u_{(0)}$ find a vector u_1° and a finite control u_2°, as well as a control v°, consistent with the estimates

$$\begin{aligned}&t_{\zeta}^{\cdot}(w^{(1)}(u_1^{\circ}), \ u_2^{\circ}, \ v) \leqslant t_{\zeta}^{\cdot\circ} \leqslant t_{\zeta}^{\cdot}(w^{(1)}(u_{(0)}), \ u, \ v^{\circ})\\ &t_{\zeta}^{\cdot\circ} = t_{\zeta}^{\cdot}(w^{(1)}(u_1^{\circ}), \ u_2^{\circ}, \ v^{\circ})\end{aligned}$$

Problem 3.3 corresponds to the control

$$\begin{aligned}&u_1^{\circ} = u^{\circ}, \quad u_2^{\circ} = 0, \quad v^{\circ} = |v^{\circ}|\, j_{\alpha,\zeta}, \quad w \in D_{1,\zeta} \cap [x_s > 0] \qquad (3.2)\\ &u_1^{\circ} = u_2^{\circ} = u^{\circ} = 0, \quad v^{\circ} = |v^{\circ}|\,(s j_{\alpha} + \sqrt{1 - s^2} j_{\beta}) \equiv v^{\circ} j_s\\ &w \in D_{1,\zeta} \cap [x_{\zeta} = 0]\\ &|v^{\circ}| = a_{\zeta} \nu d_{\zeta}^{-1}, \quad s = -\xi_{\zeta}' / \sqrt{(-x a_{\zeta} + y b_{\zeta})^2}, \quad u^{\circ} = \mu j_{\alpha\zeta}\end{aligned}$$

The solution of the Problem 3.3 relies on the relations

$$t_{\zeta}^{\cdot}\zeta_{\zeta}' = -\zeta_{\zeta}' - a_{\zeta}|v^{\circ}| + \mu^{(1)} b_{\zeta} + R_1 + R_2, \quad \zeta_{\zeta}' = \partial\zeta / \partial t\,|_{t=t_{\zeta}} \qquad (3.3)$$

$$R_1 = a_\zeta (|u| + j_{\alpha,\zeta} u) \geqslant 0 \tag{3.4}$$

$$R_2 = a_\zeta v j_{\alpha,\zeta} - (d_\zeta / 2\nu) v^2 \leqslant R_2 (w, v^\circ) = a_\zeta |v^\circ| \tag{3.5}$$

The third term in the right-hand side of Eq. (3.3) points to the equality $u_1^\circ = u^\circ$. Bound (3.4) shows that $R_1 \geqslant R_1 (w, u_2^\circ = 0) = 0$. Bound (3.5) proves the correctness of the choice of control v°. All these results are obvious in the domain $D_\zeta \cap [x_\zeta > 0, \zeta_\zeta' > 0]$. In the domain $D_\zeta \cap [x_\zeta > 0, \zeta_\zeta' = 0] \equiv D_\zeta \cap [x_\zeta > 0, t_\zeta = \tau_1]$ the derivative $t_\zeta^\cdot (w^{(1)}, u_2^\circ, v)$ exists not for all v. However the relation $t_\zeta^\cdot (w^{(1)} (u_0), u_2^\circ, v) \to -\infty$ as $w^{(1)} \to D_\zeta \cap [x_\zeta > 0, t_\zeta = \tau_1]$ is valid for any $v \neq v^\circ$. In the domain $D_\zeta \cap [x_\zeta = 0]$ the necessary conditions permit us some leeway in the choice of v° and this choice is made from the estimate

$$0 = \zeta_1^\cdot (w, u_2^\circ, v^\circ) \geqslant \zeta_1^\circ (w, u_2^\circ, v)$$

4. Let $\pi / 2 \in D_{1,t}$, i.e. the bound $\xi (w, \pi/2) = R - \nu\sqrt{\pi/2} \equiv \xi_{\pi/2} \geqslant 0$. Formula (3.3) suggests that we need to set $u^\circ = 0$ beyond the boundary $t_\zeta = \pi / 2$. The formation of the slow-action v° on the boundary $\zeta (w, \pi / 2) = 0$ of the domain $[\zeta (w, \pi / 2) < 0]$ leads to the function

$$\zeta = R - \nu d_t + \mu - x_t, \quad (w, t) \in D_{2,t} [\xi_{\pi/2} \geqslant 0, \ t \in (\pi / 2, \pi]] \tag{4.1}$$

while the successive solving of Problems 3.2, 3.3 leads to the equations

$$u_1^\circ = u_2^\circ = u^\circ = 0, \quad v^\circ = |v^\circ| j_{\alpha,\zeta}, \quad w \in D_{2,\zeta} \cap [x_\zeta > 0] \tag{4.2}$$

$$u_1^\circ = u_2^\circ = u^\circ = 0, \quad v^\circ = |v^\circ| j_s, \quad w \in D_{2,\zeta} \cap [x_\zeta = 0]$$

Beyond the boundary $\xi (w, t) = R - \nu d_t = 0$ we consider two functions $\nu_\xi (t)$, $t_\xi (\nu)$, namely, the solutions relative to ν, t of the equation $\xi = 0$, and we assume that $t_\xi (\nu) < \pi / 2$.

To construct the solution we shall argue in accordance with the scheme in [5]. Let $t_1 \in (t_\xi, \pi / 2)$ be some number and let the first player's supply of μ be so large that at $t = 0$ he can apply the impulse $u_1 = \mu_1^1 \delta$

$$\mu_1^1 = -(x b_1 / a_1 + y_\alpha) j_\alpha - y_\beta j_\beta, \quad a_1 = \sin t_1, \ b_1 = \cos t_1 \tag{4.3}$$

while for $t \in (0, t_\xi (\nu) - t_1]$ he can control by $u_1 = -v$. For $t = 0$ such a control realizes the equality $x_{t_1}^{(1)} = [(x b_1 + y_\alpha^{(1)} a_1)^2 + y_\beta^{(1)2} a_1]^{1/2} = 0$

while for $t \in (0, t_1 - t_\xi]$ it preserves the equality

$$x_{t_1-1}^{(1)} = [(x b_{t_1-t} + y_\alpha^{(1)} a_{t_1-t})^2 + y_\beta^2 a_{t_1-t}^2]^{1/2} = 0$$

and the equalities $\xi (\nu, t_\xi (\nu)) = 0$, $x_{t=t_\xi} = 0$ prove to be valid under any action taken by the second player by the instant $t^1 = t_1 - t_\xi (\nu)$, if $\mu (t^1) = \mu_\xi \geqslant 0$, i.e. the first player has a sufficient supply available. These equalities show that the inclusion $w \in D_{1,\zeta}$ is automatically realized by the instant t^1. The total expenditure of the first player's momentum at the instant t^1 is

$$|\Delta\mu| = |\mu_1^1| + \int_0^{t_1} |v| \, d\tau$$

Suppose that for fixed t_1, t_ξ the second player selects the control modulus $|v^1|$ by solving the "isoperimetric" maximum problem

$$\int_0^{t^1} |v^1|\, d\tau = \max_v \int_0^{t_1} |v|\, d\tau$$

under the condition

$$\int_0^{t^1} v^2 dt = \nu^2 - \nu_\xi^2$$

This problem's solution has the form

$$\int_0^{t^1} |v^1|\, dt = \varkappa(\nu, t_\xi, t_1) = [(\nu^2 - \nu_\xi^2)(t_1 - t_\xi)]^{1/2} \qquad (4.4)$$

$$|v^1| = \varkappa/(t_1 - t_\xi)$$

Suppose that by varying t_ξ within the limits $t_\xi \in [t_\xi\ (v),\ t_1]$ the second player selects the control $|v_1|$ from the condition that the function $\varkappa^2\ (\nu,\ t_\xi,\ t_1)$ be maximized with respect to t_ξ. Denoting $(\varkappa^2)_\xi = \partial x^2\ /\ \partial t_\xi$, we obtain the function

$$(\varkappa^2)_\xi = (-\partial \nu_\xi^2\ /\ \partial t_\xi)\ (t_1 - t_\xi) - (\nu^2 - \nu_\xi^2)$$

where

$$-\partial \nu_\xi^2 / \partial t_\xi = R^2 a_\xi^2 / d_\xi^2, \quad a_\xi = a_{t=t_\xi}, \quad d_\xi = d_{t=t_\xi}$$

It is evident that the function $(\varkappa^2)_\xi$ changes sign from plus to minus at least once as t_ξ ranges the interval $[t_\xi\ (\nu),\ t_1]$. On the other hand, the equality

$$(\varkappa^2)_{\xi,\xi} = 2\,(R^2/d_\xi^6)\, a_\xi\, [(b_\xi d_\xi^2 - a_\xi^3)(t_1 - t_\xi) - a_\xi^2 d_\xi^2]$$

is valid. The derivative of the first factor within the brackets

$$(b_\xi d_\xi^2 - a_\xi^3)_\xi' = -a(d_\xi^2 + 2a_\xi b_\xi) < 0, \quad a_\xi = \sin t_\xi, \quad b_\xi = \cos t_\xi$$

points to the bound $(\varkappa^2)_{\xi,\xi} < 0$. This means that the function $(\varkappa^\circ)_\xi$ admits of a unique and continuously differentiable zero $t_\xi^\circ\ (\nu,\ t_1)$, i.e. the point at which the function $\varkappa^2$ is maximum with respect to t_ξ

$$\varkappa^\circ(\nu, t_1) = \varkappa(\nu, t_\xi^\circ, t_1) = \max_{t_\xi} \varkappa(\nu, t_\xi, t)$$

As a result the possibility of applying control u_1 depends upon the fulfillment of the estimate

$$\zeta\ (w,\ t_1) = a_1\ (\mu - |\ \mu_1^1\ | - \varkappa^\circ\ (\nu,\ t_1)) \geqslant 0$$

and the function

$$\zeta = (\mu - \varkappa^\circ\ (\nu,\ t))\ a - x_t, \quad (w,\ t) \in D_{3,t}\ [\xi_{\pi/2} < 0, \quad t \in (t_\xi\ (\nu),\ \pi\ /\ 2]] \qquad (4.5)$$

can be continued to the function ζ in domain $D_{3,t}$. The solution of Problem 3.2 by the impulses $u = \mu_1 \delta$ under the constraint $w^{(1)} \in D_{3,\zeta}$, as well as of Problem 3.3, are

$$u_0 = mu^\circ, \quad u_1^\circ = u^\circ \qquad (4.6)$$

$$v^\circ = |v^\circ|\, j_{\alpha,\zeta}, \quad u^\circ = -|\mu_1^\circ|\, j_{\alpha,\zeta}\delta, \quad w \in D_{3,\zeta}\ [x_\zeta = 0]$$

$$v^\circ = |v^\circ|\, j_s, \quad u^\circ = u_2^\circ = -v, \quad w \in D_{3,\zeta}\ [x_\zeta = 0]$$

where the numbers $|v^\circ|$, $|\mu_1{}^\circ|$, in contrast to (3.2), are given by the equalities

$$|\mu_1{}^\circ| = x_\zeta / a_\zeta, \qquad |v^\circ| = \varkappa^1(\nu,\ t_\zeta) / (t_\zeta - t_\zeta{}^\circ(\nu,\ t_\zeta)) \tag{4.7}$$

The restriction $t_\zeta \leqslant \pi / 2$ is pointed out by the equation

$$t_\zeta{}^{\bullet}\zeta_\zeta{}' = -\zeta' - a_\zeta |v^\circ| + (\mu^1 - \varkappa_\zeta{}^\circ)\ b_\xi + R_1 + R_2 \tag{4.8}$$

where the term R_1 is of the form (3.4), while the term R_2, in contrast to (3.5), has the form

$$R_2 = a_\zeta v j_{\alpha,\zeta} - (|v^\circ| / 2)\ v^2 \leqslant R_2(w,\ v^\circ) = a_\zeta |v^\circ|$$

The construction of $u_2{}^\circ$, v° in domain $D_{3,\zeta} \cap [x_\zeta = 0]$ is a repetition of the construction in domain $D_{2,\zeta} \cap [x_\zeta = 0]$; however, v° is computed from Eq. (4.7).

5. Equation (4.8) indicates that the control u_1 is scarcely reasonable beyond the boundary $G[\xi(w,\ \pi/2) \leqslant 0,\ t_\zeta = \pi/2]$ of region $D_{3,\zeta}$. Let us assume that there $u^\circ = 0$ and construct once again the slow-action v° on the boundary $G[\xi(w, \pi/2) \leqslant 0,\ \zeta(w, \pi/2) = 0]$ of domain $D_{3,\zeta}$ from the domain $D_4\ [\xi(w,\ \pi/2) < 0,\ \zeta(w,\ \pi/2) < 0]$.

Let ν_g be some number equal to the second player's reserve at $t_\zeta = \pi/2$, and let $t_g \leqslant \pi/2 = t_\xi{}^\circ(\nu_g,\ \pi/2)$. Integration, performed by the scheme in Sect. 2, leads to the function

$$\chi = \mu - \sqrt{\nu_g{}^2 - \nu_\xi{}^2(t_g)'}\sqrt{t - t_g} - \sqrt{\nu^2 - \nu_g{}^2}\, d_{t-\pi/2} - x_t$$

$$d_{t-\pi/2} = \sqrt{(t - \pi/2)/2 - (\sin 2t)/4}$$

The functions $\nu_g{}^2$, t_g are as yet unknown. However, the necessary conditions point out that the absolute value $|v^\circ|$ of the optimal control is continuous on the boundary G. This solution yields the equality

$$|v^\circ|^2 = (\nu_g{}^2 - \nu_\xi{}^2(t_g))/(t - t_g) = (\nu^2 - \nu_g{}^2)/d_{t-\pi/2}$$

Eliminating the function $\nu_g{}^2$ from this equality and replacing t_g by t_ξ, we obtain the function

$$\chi_1 = \mu - (\nu^2 - \nu_\xi{}^2(t_\xi))^{1/2}(\pi/2 - t_\xi + d_{t-\pi/2}^2)^{1/2} - x_t$$

We continue the function $\varkappa(\nu,\ t_\xi,\ t)$ into the region $[t_\xi < \pi/2,\ t > \pi/2]$ by the formula

$$\varkappa^2 = (\nu^2 - \nu_\xi(t_\xi))(\pi/2 - t_\xi + d_{t-\pi/2}^2)$$

The operation $\max_{t_\xi \leqslant \pi/2} \varkappa^2$ leads to the function $(\varkappa^2)_\xi = (-\partial\nu_\xi{}^2 / \partial t_\xi)(\pi/2 - t_\xi + d_{t-\pi/2}^2) - (\nu^2 - \nu_\xi{}^2)$ and continues the function $t_\xi{}^\circ(\nu,\ t)$ as a zero of the function $(\varkappa^2)_\xi$ into the domain

$$D_{4,t}[\xi(w, \pi/2) < 0, (\varkappa^2)_{\pi/2}' \leqslant 0, t \in (\pi/2, 3\pi/2)]$$

$$\varkappa_{\pi/2}^{2'} = \partial x^2(\nu, t_\xi = \pi/2, t > \pi/2)/\partial t_\xi$$

In the domain

$$D_{5,t}[\xi(w,\ \pi/2) < 0,\ (\varkappa^2)_{\pi/2}' > 0,\ t \in (\pi/2,\ 3\pi/2)]$$

the function $t_\xi{}^\circ(\nu,\ t)$ is continued by the equality $t_\xi{}^\circ = \pi/2$. All the constructions listed lead to the function

$$\begin{aligned}
&\zeta = \mu - \varkappa^\circ(\nu, t) - x_t, \quad (w, t) \in D_{4,t} \bigcup D_{5,t} \qquad (5.1)\\
&(\varkappa^\circ)^2 = (\nu^2 - \nu_\xi(t_\xi^\circ))(\pi / 2 - t_\xi^\circ + d_{t-\pi/2}^2), \quad (w, t) \in D_{4,t}\\
&(\varkappa^\circ)^2 = (\nu^2 - 4R^2 / \pi)\, d_{t-\pi/2}^2, \quad (w, t) \in D_{5,t}
\end{aligned}$$

Problems 3.2 and 3.3 have solutions of the form

$$\begin{aligned}
&u_1^\circ = u_2^\circ = u^\circ = 0 \qquad (5.2)\\
&v^\circ = |v^\circ| j_{\alpha,\zeta}\\
&v^\circ = -|v^\circ| j_{\alpha,\zeta}, \qquad w \in [D_{4,\zeta} \bigcup D_{5,\zeta}] \bigcap [x_\zeta > 0]\\
&v^\circ = |v^\circ| j_s\\
&|v^\circ| = |v^\circ| j_{-s}, \qquad w \in [D_{4,\zeta} \bigcup D_{5,\zeta}] \bigcap [x_\zeta = 0]
\end{aligned}$$

Here

$$\begin{aligned}
&|v^\circ| = \varkappa_\zeta^\circ / (\pi / 2 - t_\xi^\circ(\nu, t_\zeta) - d_{\pi/2} - t_\zeta), \quad w \in D_{4,\zeta}\\
&|v^\circ| = \varkappa_{\pi/2}^\circ / d_{\pi/2 - t_\zeta}, \quad w \in D_{5,\zeta}\\
&x_\zeta^\circ = \varkappa^\circ(\nu, t_\zeta), \quad \varkappa_{\pi/2}^\circ = \varkappa(\nu, \pi / 2, t_\zeta)\\
&j_{-s} = -s j_\alpha + \sqrt{1 - s^2}\, j_\beta
\end{aligned}$$

6. When $\omega^2 = 0$ system (1.1) loses the term in $-x$ in the second equation. The necessity of norming the time drops out. We present a brief summary of the results for this simple case $\zeta = R - \nu\sqrt{t^3 / 3} + \mu t - x_t, \quad (w, t) \in D_{1,t}\, [\xi(\omega, t) = R - \nu\sqrt{t^3 / 3} \geqslant 0]$

$$\begin{aligned}
&\zeta = (\mu - \varkappa^\circ(\nu, t))\, t - x_t, \quad (w, t) \in D_{2,t} \quad [\xi(w, t) < 0]\\
&\varkappa^2(\nu, t_\xi, t) = (\nu^2 - 3R^2 / t_\xi^3)(t - t_\xi)\\
&(\varkappa^2)_\xi = 9R^2 (t - t_\xi) / t_\xi^4 - (\nu^2 - 3R^2 / t_\xi^2)\\
&\varkappa^\circ(\nu, t) = \varkappa(\nu, t_\xi^\circ(\nu, t), t), \quad x_t = \sqrt{(x - y_\alpha t)^2 + y_\beta^2 t^2}
\end{aligned}$$

As above, the function $t_\xi^\circ(\nu, t)$ is a zero of the function $(\varkappa^2)_\xi$. The solutions of Problems 3.2, 3.3 have the form

$$\begin{aligned}
&u_1^\circ = u^\circ = -\mu j_{\alpha,\zeta}, \quad u_2^\circ = 0, \quad v^\circ = |v^\circ| j_{\alpha,\zeta}, \quad w \in D_{1,\zeta} \bigcap [x_\zeta > 0]\\
&u_1^\circ = u^\circ = u_2^\circ = 0, \quad v^\circ = |v^\circ| j_s, \quad w \in D_{1,\zeta} \in [x_\zeta = 0]\\
&|v^\circ| = t_\zeta \nu / \sqrt{t_\zeta^3 / 3}, \quad j_\alpha, \quad w \in D_{1,\zeta}\\
&u_1^\circ = u^\circ = -(x_\zeta / t_\zeta) j_{\alpha,\zeta}, \quad u_2^\circ = 0, \quad v^\circ = |v^\circ| j_{\alpha,\zeta}, \quad w \in D_{2,\zeta} \bigcap [x_\zeta > 0]\\
&u_1^\circ = 0, \quad u_2^\circ = u^\circ = -v, \quad v^\circ = |v^\circ| j_s, \quad w \in D_{2,\zeta} \bigcap [x_\zeta = 0]
\end{aligned}$$

where the modulus $|v^\circ|$ is given by the equality

$$|v^\circ| = \varkappa^\circ(\nu, t) / (t - t_\xi^\circ(\nu, t)), \quad w \in D_{2,\zeta}$$

7. The controls u°, v° generate a time-varying vector x_1° with projections x_α°, x_β°. Let us give a brief geometric description of this. Let $w \in D_{2,4,5,\zeta}$. We fix $j_{\alpha,\beta,\gamma}$; then to make the position fall onto the boundary of domains $D_{1,3}$ the vector x_1° is varied in accord with the equation

$$x_\alpha = xb_t + y_\alpha a_t + x_{\alpha,q}, \quad x_\beta = y_\beta a_t + x_{\beta,q}$$
$$x_{\alpha,q} = (j_{\alpha,\zeta})_\alpha q, \quad x_{\beta,q} = (j_{\alpha,\zeta})_\beta q$$
$$q = |v^\circ| \int_0^t \sin(t-\tau)\sin(t_\zeta - \tau)\,d\tau$$

Here $(j_{\alpha,\zeta})_{\alpha,\beta}$ are the projections of vector $j_{\alpha,\zeta}$ on the unit vectors $j_{\alpha,\beta}$. By stretching the point somewhat we can assume that the representative point moves along an ellipse with a moving center $O\ (x_{\alpha,q}, x_{\beta,q})$. The orientation and axis magnitudes of this ellipse are constant. If the position $w \in D_{1,\zeta}$, the motion from it unfolds for $t > 0$ by the equations

$$x_\alpha^\circ = xb_t + y_\alpha^{(1)} a_t + x_{\alpha,q}, \quad x_\beta^\circ = y_\beta^{(1)} + x_{\beta,q}$$

$$y_\alpha^{(1)} = y_\alpha + u_\alpha^\circ, \qquad y_\beta^{(1)} = y_\beta + u_\beta^\circ$$

since when $w \in D_{1,\zeta}$ the first player starts to control with the impulse $u^\circ = -\mu j_{\alpha,\zeta}$.

If $w \in D_{3,\zeta}$, then the impulse $u^\circ = -(x_\zeta / a_\zeta) j_{\alpha,\zeta}$ leads to the equality $y_\beta^{(1)} = 0$, while the equality $u^\circ + v^\circ = 0$ shows that up to falling onto the boundary of domain $D_{1,\zeta}$ the motion is rectilinear and follows the equation

$$x_\alpha^\circ = xb_t + y_\alpha^{(1)} a_t, \quad x_\beta^\circ = 0$$

The equalities $\mu = \xi(w, t_\zeta) = x_\zeta = 0$ must be fulfilled on the boundary of domains $D_{3,\zeta}$ and $D_{1,\zeta}$ After intersection with this boundary the motion proceeds in accord with the equations

$$x_\alpha^\circ = xb_t + y_\alpha a_t + (j_s)_\alpha q, \qquad x_\beta^\circ = (j_s)_\beta q$$

Here the values x, y_α, j_s, t_ζ take on the boundary of domains $D_{3,\zeta}$, $D_{1,\zeta}$ and the time t is counted off from the instant of falling onto the boundary. Motion takes place along an ellipse with a biased center. It can be shown that when $t = t_\zeta$ the trajectory indicated is tangent to the sphere $x = R$.

8. Further analysis meets with one essential difficulty which consists in the following. It can be shown that with respect to the variable t the function ζ admits of no more than three isolated maxima ζ_1, ζ_2, ζ_3 on the interval $t \in (0, 3\pi/2)$ at the points τ_1, τ_2, τ_3, respectively. The last maximum is not essential for the analysis of the structure of function t_ζ since the estimate $\zeta_2 > \zeta_3$ is valid. However, it is geometrically obvious that on the set $F\,[\zeta_1 = 0,\ \zeta_2 \geqslant 0]$ the function t_ζ undergoes a positive jump as it passes from the side of $F_1[\zeta_1 > 0,\ \zeta_2 \geqslant 0]$ into the domain $F_2[\zeta_1 < 0,\ \zeta_2 \geqslant 0]$. In fact, the equality $t_\zeta = \tau_1$ is valid on set F , while the estimate $\tau_1 < t_\zeta < \tau_2$ is valid for $w \in F_2$. By $t_{\zeta(2)}$ we denote the second-by-count zero of function ζ. The second player employs $v^\circ(w)$ at the positions close to set F, lying in domain F_2 . If a control $u(w)$ such that $\zeta_1^{\cdot}(w, u(w), v^\circ(w)) > 0$ exists, then the position can hit onto set F. It can be shown that the maximum value of $\zeta_1^{\cdot}(w, u, v^\circ)$ is realized at $u = 0$. Thus, the bound $\zeta_1^{\cdot}(w, 0, v^\circ(w)) > 0$ points up the risk of falling onto set F.

Unfortunately, we have not succeeded in resolving effectively the question of the existence of positions satisfying the estimate

$$\lim \zeta_1^{\cdot}(w \to F,\ 0,\ v^\circ(w \to F)) \geqslant 0$$

It can be shown that in the domain $D_{3,\zeta} \cap F$ the answer to this question is equivalent to the existence of positions satisfying the estimate

$$\varphi = -(\mu - \varkappa^\circ(w, \tau_1))\, b_{\tau_1} + a_{\tau_1} \mid v^\circ(w, \tau_1) \mid - a_2 \mid v_2^\circ \mid \geqslant 0$$

$$a_2 = \sin t_{\zeta(2)}, \quad \mid v_2^\circ \mid = \mid v^\circ(w, t_\zeta = t_{\zeta(2)}) \mid, \quad a_{\tau_1} = \sin \tau_1, \quad b_{\tau_1} = \cos \tau_1$$

while in the domain $D_{1,\zeta} \cap F$ the question resolves the estimate

$$\varphi(w) = -\mu\, b_{\tau_1} + a_{\tau_1} \mid v^\circ(w, \tau_1) \mid - a_2 \mid v_2^\circ \mid \geqslant 0$$

The set $H\,[F \cap [\varphi(w) \geqslant 0]]$ is defined in the domain $[x_{\tau_1} > 0,\ x_{t=t_{\zeta(2)}} > 0]$ by the two equations $\zeta(w, \tau_1) = \zeta(w, t_{\zeta(2)}) = \zeta_t'(w, \tau_1) = 0$ and by the two estimates $\varphi(v) \geqslant 0$, $t_{\zeta(2)} \leqslant \tau_2$. Unfortunately, we do not even know whether the set H is empty or not. If set H is not empty, then by H_1 we denote the domain occupied by the trajectories w° intersecting domain H in due course, $t_\zeta(w) > t_{\zeta(2)}$ $(w \in H)$. We denote the remaining domains $W^\circ\,[\zeta(w, 0) < 0,\ \max_{t \leqslant 3\pi/2} \zeta(w, t) \geqslant 0]$ by H_2. By H_3 we denote the domain defined by the estimates $H_3[0 > \zeta(w, 0) = \zeta_1 = \zeta_2,\ \ \varphi_1(w) > 0]$, where $\varphi_1(w)$ is obtained from $\varphi(w)$ by replacing $t_{\zeta(2)}$ with τ_2. In these terms we state the results of the investigation without cumbersome proofs.

8.1. When $w \in W^\circ$ the controls u°, v° realize the time t_ζ of first hitting onto set M and the second player cannot make the motion onto M late by using the pair u°, v.

8.2. When $w \in H_2$ the first player cannot lessen the time t_ζ, i.e. cannot lead the position onto M earlier than by the instant t_ζ by using any pair u, v° retaining the trajectory in domain H_2.

8.3. If the sets H_1, H_3 are empty, the control

$$v_0(w) = v^\circ(\tau_1), \quad w \in W_{1,0}$$
$$v_0(w) = v^\circ(\tau_2), \quad w \in W_{2,0}$$
$$v_0(w) = 0, \quad w \in W_{3,0}$$

does not permit the first player to lead the motion onto M if this motion starts outside the domain $W^\circ \cup [x \leqslant R]$ or leaves this domain under nonoptimal actions of the first player.

The control $v^\circ(\tau_1)$ $(v^\circ(\tau_2))$ is the control $v^\circ(w, t_\zeta)$ after t_ζ is replaced by τ_1 (τ_2), respectively. Domain $W_{1,0}$ combines the positions admitting of τ_1, τ_2 with estimates $0 > \zeta_0 = \zeta(w, 0) \leqslant \zeta_1$, $\zeta_1 > \zeta_2$, as well as the positions not admitting of τ_2 but admitting of τ_1 with bounds $0 > \zeta_0 \leqslant \zeta_1$. In domain $W_{2,0}$ occur the positions admitting of τ_2 with estimates $0 > \zeta_0 \leqslant \zeta_1 \leqslant \zeta_2$. The remaining positions occur in domain $W_{3,0}$

$$W_{3,0}\,[W \diagdown [[x \leqslant R] \cup W^\circ \cup W_{1,0} \cup W_{2,0}]]$$

The geometric meaning of the difficulty mentioned is simple. Let $w \in [W^\circ \cap \zeta_1 < 0,\ \zeta_2 > 0]$. Here the second player's control v° is constructed from the condition $t_\zeta^{\cdot}(w, u, v^\circ) \geqslant t_\zeta^{\cdot}(w, u, v)$. If the first player employs $u \neq u^\circ$, then ζ_1 can increase up to zero, and t_ζ receives a negative jump. When $w \in W_0^{\cdot}\,[W \diagdown [W^\circ \cup [x < R]] \cap [\zeta_1 \subset \zeta_2]$ the second player constructs v_0 from the condition $\zeta_2^{\cdot}(w, u, v^\circ) \leqslant \zeta_2^{\cdot}(w, u, v)$. If the first player employs $u \neq u^\circ$, the equality $\zeta_1 = \zeta_2$ can be realized. The second player is in difficulty at these positions. If he should try to preserve or lessen ζ_1 (ζ_2), then there is no guarantee that the first player would apply the control u'' so that ζ_2 (ζ_1) would increase and, as a result, the position hit onto the set $\zeta_2 = 0$ ($\zeta_1 = 0$).

The author has not succeeded in resolving the question of the existence of controls of the first player increasing the lesser maximum. Therefore, the theorem contains reservations. The difficulty described is sufficiently typical. Its existence and ways for overcoming it were noted in [2].

REFERENCES

1. Isaacs, R., Differential Games. New York, J. Wiley and Sons, Inc., 1965.
2. Krasovskii, N. N., Game Problems on the Contact of Motions. Moscow, "Nauka", 1970.
3. Pozharitskii, G. K., Game problem of the hard contact of two points with an impulse thrust in a linear central field. PMM Vol. 36, № 6, 1972.
4. Pozharitskii, G. K., Impulse tracking of a point with bounded thrust. PMM Vol. 37, № 2, 1973.
5. Pozharitskii, G. K., Generalized problem of the impulse pursuit of a point with bounded thrust. PMM Vol. 38, № 1, 1974.
6. Pozharitskii, G. K., Game problem of impulse "hard" contact in a position attraction field with an opponent who realizes a bounded thrust. PMM Vol. 39, № 2, 1975.

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ASYMPTOTIC SOLUTION OF CERTAIN PROBLEMS OF TIME-OPTIMAL TYPE

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We examine the optimal control problem for a system in which the process termination time is not fixed. The system of equations of motion contains a small parameter and is reduced to the form of systems with rotating phase. We assume that the frequency depends upon "slow time", while the control occurs in the small terms, so that the system is weakly controllable. Using the averaging method we construct a solution of the optimal control problem and we assume that the time interval over which the process evolves is a quantity of the order of $1 / \varepsilon$, where ε is the small parameter. This assumption proves to be a natural one if the terminal manifold depends only on the slow variables. Thus, we investigate the cases, of interest in practice, of controlled systems with small but protracted controls. We solve certain concrete problems with the use of the canonic averaging method developed.

1. Statement of the problem. Let the system's motion be described by the equations

$$a^{\cdot} = \varepsilon f(\tau, a, \psi, u, \varepsilon), \quad a(t_0) = a_0 \tag{1.1}$$

$$\psi^{\cdot} = \nu(\tau) + \varepsilon F(\tau, a, \psi, u, \varepsilon), \quad \psi(t_0) = \psi_0$$